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Stability of the Langdon–Dawson Advective Algorithm*

A recent article by Godfrey [1] investigates the numerical stability of one-dimensional electromagnetic PIC-CIC plasma simulation algorithms [2, 3], with emphasis on the numerical Cherenkov instability [4, 5]. In [1, Sect. 6], an advective differencing scheme incorrectly attributed to Langdon was discussed. This note clarifies part of that discussion, analyzes the actual Langdon–Dawson one-dimensional algorithm [6], and suggests a corresponding multidimensional differencing scheme.

The basic approach of advective differencing is to integrate numerically Maxwell's equations along their vacuum characteristics. This is straightforward in one dimension; where right- and left-going ($\pm$) transverse waves explicitly decouple, and leads to

$$(E_y \pm B_z)_{m+1,n\pm 1} = (E_y \pm B_z)_{m,n} - (J_y)^{\pm}_{m+1/2,n\pm 1/2}\,\Delta t, \tag{1}$$

with a similar equation for E_z and B_y. The integer subscripts m and n designate time and space, respectively. Note that Eq. (1) requires $\Delta x = \Delta t$. (Units are chosen such that the speed of light and the plasma frequency each equal one.)

It can be shown that the improved stability associated with advective differencing schemes is due not so much to the dispersionless vacuum transport of the fields, per se, as to the less conventional methods of determining the mesh currents usually employed with advective differencing. Thus, for the case considered in [1], J^+ and J^- are equal, defined in the notation of [5] as

$$J_{m+1/2,n\pm 1/2} = \int dt\,dx\,j(t,x)\,\tfrac{1}{2}[\delta(t - t_{m+1-\epsilon}) + \delta(t - t_{m+\epsilon})]\,S(x - x_{n\pm 1/2}) \tag{2a}$$

in the limit of vanishing $\epsilon > 0$, or more explicity,

$$J_{m+1/2,n\pm 1/2} = \sum_i V_{i,m+1/2}\cdot \tfrac{1}{2}[S(X_{i,m+1} - x_{n\pm 1/2}) + S(X_{i,m} - x_{n\pm 1/2})]. \tag{2b}$$

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In other words, currents are interpolated onto the mesh with particle positions at times t_{m+1} and t_m, but with velocities from $t_{m+1/2}$, and then averaged to give $J_{m+1/2}$. In Eq. (2), $S(x)$ is a spatial interpolation function, while $j(t, x)$ is the particle current; X_i and V_i are the position and velocity of particle i. The principal effect of so defining J is to smooth the current term in the dispersion relation for Eq. (1) by the velocity dependent factor $\cos(kv\,\Delta t/2)$. For v large the factor suppresses nonphysical effects for k near $\pm\pi/\Delta x$.

On the other hand, it also distorts physical phenomena in this region of wave-number space. This shortcoming is, however, overstated in [1]. For any algorithm, and not for this one only, caution must be exercised in the interpretation of the behavior of large k modes. This definition of J is successfully employed in electromagnetic codes at the Lawrence Livermore Laboratory [6] and the Naval Research Laboratory [7].

The differencing scheme actually developed by Langdon and Dawson [6] defines mesh current not as in Eq. (2), but as

$$J^{\pm}_{m+1/2,n+1/2} = \int dt\,dx\,j(t, x) \cdot \tfrac{1}{2}[\delta(t - t_{m+1-\epsilon})\,S(x - x_{n\pm1}) + \delta(t - t_{m+\epsilon})\,S(x - x_n)], \quad (3a)$$

or

$$J^{\pm}_{m+1/2,n\pm1/2} = \sum_i V_{i,m+1/2}\,\tfrac{1}{2}[S(X_{i,m+1} - x_{n\pm1}) + S(X_{i,m} - x_n)]. \quad (3b)$$

Current is averaged along vacuum characteristics rather than at fixed points in space.

The corresponding dispersion relation for the single-species cold-beam problem analyzed in [1] reads

$$\begin{aligned}
&\sin^2(\omega\,\Delta t/2) - \sin^2(k\,\Delta t/2)\\
&\quad = |\,S(k)|^2\,(\Delta t/2)^2 \csc[(\omega - kv)\,\Delta t/2]\,\{\cos(k\,\Delta t/2)\cos(kv\,\Delta t/2)\\
&\qquad \times [\sin(\omega\,\Delta t/2)\cos(k\,\Delta t/2) - v\cos(\omega\,\Delta t/2)\sin(k\,\Delta t/2)]\\
&\qquad + \sin(k\,\Delta t/2)\sin(kv\,\Delta t/2)\\
&\qquad \times [\sin(k\,\Delta t/2)\cos(\omega\,\Delta t/2) - v\sin(\omega\,\Delta t/2)\cos(k\,\Delta t/2)]\}. \qquad (4)
\end{aligned}$$

For small particle velocities the right side of Eq. (4) reduces to the familiar expression characteristic of many differencing schemes [8] multiplied by the factor $\cos(k\,\Delta t/2)$. Any numerical problems resulting from the tangency of light curves near $k = \pm\pi/\Delta x$ are, therefore, virtually eliminated. Moreover, as v increases, the light curves actually more apart slightly.

This can be seen most clearly for $v = \pm 1$, when Eq. (4) becomes

$$\sin^2(\omega \, \Delta t/2) - \sin^2(k \, \Delta t/2) = | S(k)|^2 (\Delta t/2)^2 \cos(k \, \Delta t). \tag{5}$$

The plasma reactance is, in effect, negative for $| k | > \pi/2\Delta x$, although the term is so small in this region that it scarcely affects simulation accuracy. That the phase velocity of the light waves falls below one, does, however, suggest the possibility of a numerical Cherenkov instability. Fortunately, the sign of the coupling term between the spurious beaming mode [1] and the nearby light mode is positive, and no instability occurs. Figure 1 illustrates the case $\Delta x = \Delta t = 0.5$ and $v = 0.95$. Frequencies are purely real.

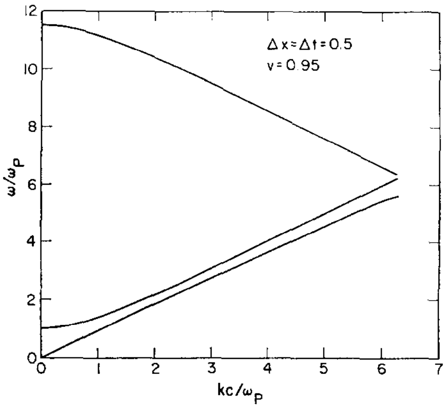

FIG. 1. Solution of Eq. (4) in the range $0 < \omega < 2\pi/\Delta t$, $0 < k < \pi/\Delta x$ for $\Delta x = \Delta t = 0.5$ and $v = 0.95$.

It is instructive to recast Eqs. (1) and (3) as

$$\begin{aligned}
(E_y)_{m+1,n} &= \tfrac{1}{2}[(E_y)_{m,n+1} + (E_y)_{m,n-1} - (B_z)_{m,n+1} + (B_z)_{m,n-1}] \\
&\quad - \tfrac{1}{4}[2(J_y)_{m+1-\epsilon,n} + (J_y)_{m+\epsilon,n+1} + (J_y)_{m+\epsilon,n-1}] \, \Delta t, \\
(B_z)_{m+1,n} &= \tfrac{1}{2}[(B_z)_{m,n+1} + (B_z)_{m,n-1} - (E_y)_{m,n+1} + (E_y)_{m,n-1}] \\
&\quad - \tfrac{1}{4}[-(J_y)_{m+\epsilon,n+1} + (J_y)_{m+\epsilon,n-1}] \, \Delta t,
\end{aligned} \tag{6}$$

where

$$J_{m\pm\epsilon,n} = \int dt \, dx \, j(t, x) \, \delta(t - t_{m\pm\epsilon}) \, S(x - x_n) \tag{7a}$$

or

$$J_{m\pm\epsilon,n} = \sum_i V_{i,m\pm 1/2} S(X_{i,m} - x_n). \tag{7b}$$

If Eq. (6) is Fourier transformed in space, the resulting equations are readily generalized to higher dimensions,

$$\begin{aligned} \mathbf{E}_{m+1} &= \mathbf{E}_m \cos(k\,\Delta t) + i\hat{\mathbf{k}} \times \mathbf{B}_m \sin(k\,\Delta t) \\ &\quad - \tfrac{1}{2}[\mathbf{J}_{m+1-\epsilon} + \mathbf{J}_{m+\epsilon} \cos(k\,\Delta t)]\,\Delta t \\ \mathbf{B}_{m+1} &= \mathbf{B}_m \cos(k\,\Delta t) - i\hat{\mathbf{k}} \times \mathbf{E}_m \sin(k\,\Delta t) \\ &\quad + \tfrac{1}{2} i\hat{\mathbf{k}} \times \mathbf{J}_{m+\epsilon} \sin(k\,\Delta t)\,\Delta t. \end{aligned} \tag{8}$$

Here, $\hat{\mathbf{k}}$ is a unit vector along $\mathbf{k}$. By one means or another, $\mathbf{k} \cdot \mathbf{E} = -i\rho$ also must be enforced [5, 7, 9]. Solving Maxwell's equations by Fourier transform methods already has been demonstrated as practical [10, 11]. The only real disadvantage of Eq. (8) is the necessity of determining and separately storing $\mathbf{J}$ twice per time step. (It may also be possible to extend Eq. (6) to higher dimensions without recourse to Fourier transforms. We have not yet pursued this idea.)

To avoid the extra storage requirements, one may wish to employ Eq. (2) instead of Eq. (3). The new finite difference equations,

$$\begin{aligned} \mathbf{E}_{m+1} &= \mathbf{E}_m \cos(k\,\Delta t) + i\hat{\mathbf{k}} \times \mathbf{B}_m \sin(k\,\Delta t) - \mathbf{J}_{m+1/2} \cos(k\,\Delta t/2)\,\Delta t, \\ \mathbf{B}_{m+1} &= \mathbf{B}_m \cos(k\,\Delta t) - i\hat{\mathbf{k}} \times \mathbf{E}_m \sin(k\,\Delta t) + i\hat{\mathbf{k}} \times \mathbf{J}_{m+1/2} \sin(k\,\Delta t/2)\,\Delta t, \end{aligned} \tag{9}$$

are strikingly similar to those of J. P. Boris *et al.* [10].

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